COMBINATORICA
Akadémiai Kiadó — Springer-Verlag

COMBINATORICA 9 (2) 1989 133—143

SURVIVAL TIME OF A RANDOM GRAPH

A. M. FRIEZE

Received September 3, 1987

Let $V_n=\{1,2,...,n\}$ and $e_1, e_2, ..., e_N$, $N=\binom{n}{2}$ be a random permutation of $V_n^{(2)}$. Let $E_t=\{e_1, e_2, ..., e_t\}$ and $G_t=(V_n, E_t)$. If Π is a monotone graph property then the hitting time $\tau(\Pi)$ for Π is defined by $\tau=\tau(\Pi)=\min\{t\colon G_t\in\Pi\}$. Suppose now that G_τ starts to deteriorate i.e. loses edges in order of *age*, $e_1, e_2, \ldots$. We introduce the idea of the *survival* time $\tau=\tau'(\Pi)$ defined by

$$\tau' = \max\{u\colon (V_n, \{e_u, e_{u+1}, ..., e_T\})\in\Pi\}.$$

We study in particular the case where Π is k-connectivity. We show that

1) $\lim_{n\to\infty} \Pr(\tau'\geqq an)=e^{-2a}$ for $a\in\mathbf{R}^+$

2) $\lim_{n\to\infty} \frac{1}{n} E(\tau')=\frac{1}{n}$

i.e. τ'/n is asymptotically negative exponentially distributed with mean $\frac{1}{2}$.

1. Introduction

Let $V_n=\{1,2,...,n\}$ and $e_1, e_2, ..., e_N$, $N=\binom{n}{2}$ be a random permutation of $V_n^{(2)}$, the edge of set of the complete graph K_n. If $G_t=(V_n, E_t)$ where $E_t=\{e_1, e_2, ..., e_t\}$ then the Markov chain $\mathscr{G}=(G_t)_{t=0}^N$ is the central object of study in the theory of random graphs. If Π is a monotone increasing graph property then one wishes to establish asymptotic properties of the distribution of the *hitting time*

$$\tau(\Pi)=\min\{t\colon G_t\in\Pi\}.$$

Erdős and Rényi [4], [5] showed this to be an interesting problem and hundreds of papers have been written on this subject since the early papers — see the recent encyclopaedic text of Bollobás [2] or the introductory text of Palmer [7]. Erdős and Rényi thought of $\mathscr{G}$ as describing the *evolution* of some *living organism*. Suppose we pursue this analogy and think of $\tau(\Pi)$ as the time when this organism is *fully grown*. Going the way of all flesh our graph will start to *deteriorate*, say by

AMS subject classification (1980); 05 C 80

losing edges. We will assume that older edges disappear first. We propose to study the progress of the graph

$$H_u = (V_n, \{e_u, e_{u+1}, \dots, e_\tau\})$$

for various properties Π. We define the *survival time* $\tau(\Pi)$ for the process $\mathscr{G}$ by

$$\tau'(\Pi) = \max\{u\colon\ H_u \in \Pi\}.$$

The property discussed in this paper is k-connectivity. One can imagine many other properties worthy of study and we list some open problems at the end of the paper.

Our main result can be expressed as follows: let $k \geqq 1$ be a fixed integer. Let D_k denote the property of having minimum degree at least k and let C_k denote the property of being k-connected

Theorem

If Π is C_k or D_k then

(i) $\lim_{n\to\infty} \Pr(\tau'(\Pi) \geqq an) = e^{-2a}$ *for* $a \in \mathbf{R}^+$

(ii) $\lim_{n\to\infty} \frac{1}{n} E(\tau'(\Pi)) = \frac{1}{2}$

(iii) $\lim_{n\to\infty} \Pr(\tau'(C_k) = \tau'(D_k)) = 1.$ ▮

2. Preliminaries

We need some probability inequalities related to the k-connectivity of a random graph. The main thrust of the assertions are from Erdős and Rényi [6] but the calculations giving the precise bounds used are left to an appendix.

Let $Z_m^{(r)}$ be the set of vertices of degree r in G_m and let $z_m^{(r)} = |Z_m^{(r)}|$.

Let $S \subseteqq V_n$ be a *non-trivial* separator of G_m if $G_m[V_n - S]$ is not connected but has no isolated vertices.

Lemma 1

Let

$$m = \frac{1}{2} n \log n + \frac{1}{2}(k-1) n \log\log n - \frac{1}{2}\omega n \quad \text{be integer.} \tag{2.1}$$

(a) *Suppose that in* (2.1) $\omega = \omega(n) \to \infty$ *but* $\omega = o(\log\log n)$. *In what follows* $\alpha = \alpha(k)$ *is some 'constant' (depending only on the constant k) whose exact value is unimportant.*

$$\Pr(\delta(G_m) < k-1) \leqq \frac{\alpha e^{\omega}}{\log n} \tag{2.2a}$$

where $\delta(G)$ denotes the minimum degree of graph G.

$$\Pr(\delta(G_m) \geqq k) \leqq \alpha e^{-\omega} \tag{2.2b}$$

$$\Pr\left(\left|z_m^{(k-1)} - \frac{e^{\omega}}{(k-1)!}\right| \geqq \frac{\varepsilon e^{\omega}}{(k-1)!}\right) \leqq \frac{\alpha e^{-\omega}}{\varepsilon^2} \quad \text{for} \quad 0 < \varepsilon < 1 \tag{2.2c}$$

(2.2d) $\Pr$ (*G_m has a non-trivial separator of size* $\leqq k-1)\leqq\alpha\dfrac{(\log n)^2}{\sqrt{n}}$

(2.2e) $\Pr$ ($\exists\omega$: $|\omega|\leqq\log\log n$, *m(as in* (2.1)) *such that G_m has a non-trivial separator S of size $\leqq k-1$ for which $G_m[V_n-S]$ has a component of size t,* $3\leqq t\leqq\frac{1}{2}|V_n-S|)=O\left(\dfrac{(\log n)^2}{\sqrt{n}}\right)$.

(2.2f) $\Pr$ ($\exists v, w\in Z_m^{(k-1)}$: *the distance from v to w in G_m is 2 or less*)$\leqq\alpha\sqrt{\dfrac{\log n}{n}}$.

(b) *If we relax the restriction* $\omega=o(\log\log n)$ *in* (a) *to* $\omega n\leqq m$ *then we can still prove*

$$\Pr(\delta(G_m)\geqq k)\leqq\alpha e^{-\omega/2}. \tag{2.3}$$

(c) *Let now* $m=\frac{1}{2}n\log n+\frac{1}{2}(k-1)n\log\log n+\frac{1}{2}\omega n$ *where* $\omega=\omega(n)\to\infty$. *Then*

$$\Pr(\delta(G_m)<k)\leqq\max\{\alpha e^{-\omega}, n^{-2}\}. \tag{2.4}$$ ▮

The hitting time of C_k was discussed by Bollobás and Thomason [3]. That paper established

$$\lim_{n\to\infty}\Pr(\tau(C_k)=\tau(D_k))=1. \tag{2.5}$$

Let $m_1=\left[\frac{1}{2}n\log n+\frac{1}{2}(k-1)n\log\log n-\frac{1}{2}n\log\log\log n\right]$ and $m^*=\tau(D_k)$. It will be useful to think of G_m^* in the following terms (Bollobás [1]):

$$E_{m^*}=E_{m_1}\cup X\cup Y$$

where

$$X=\{e\in E_{m^*}-E_{m_1}:\ e\cap Z_{m_1}^{(k-1)}=\emptyset\}$$

and

$$Y=E_{m^*}-(E_{m_1}\cup X).$$

This is valid, conditional on an event of probability $1-O((\log\log n)^{-1})$ — see (2.2b) with $m=m_1$ and $\omega=\log\log\log n$.

Now, in this case,

$$X\text{ is a random }|X|\text{-subset of }(V_n-Z_{m_1}^{(k-1)})^{(2)}. \tag{2.6}$$

Also

$$\Pr(|X\cup Y|>n\log\log\log n)\leqq\Pr(\delta(G_m)\leqq k-1)\leqq\frac{\alpha}{\log\log n}. \tag{2.7}$$

where $m_2=m_1+[n\log\log\log n]$, by (2.4). Furthermore

$$\Pr(\exists e\in Y:\ e\subset Z_{m_1}^{(k-1)})\leqq\frac{\log n}{n} \tag{2.8}$$

and

$$\Pr(\exists z\in Z_{m_1}^{(k-1)}:\ |\{e\in Y:\ z\in e\}|>6\log\log\log n)\leqq\frac{1}{\log\log n}. \tag{2.9}$$

Given $|X\cup Y|\leqq n\log\log\log n$ and $|Z_{m_1}^{(k-1)}|=O(\log\log n)$ (use $\varepsilon=1$ in (2.2c)) it is unlikely that the conditions in (2.8), (2.9) will be violated — we are after all adding at most $n\log\log\log n$ random edges.

3. Proof of the Theorem

(i) and (iii).

Let $a\in\mathbf{R}^+$ and $u=[an]$. We show first that

$$\lim_{n\to\infty}\Pr\big(\delta(H_u)=k\big)=e^{-2a}. \tag{3.1}$$

We aim in fact to prove

$$\big|\Pr\big(\delta(H_u)=k\big)-e^{-2a}\big|=O\left(\frac{(\log\log\log n)^2}{\log\log n}\right) \tag{3.2a}$$

where the hidden constant in the "big O" notation may depend on k. Thus from now on, if after describing an event $\mathscr{E}$ we write $[P=1-o(1)]$, we mean $\Pr(\mathscr{E})=1-O\big((\log\log\log n)^2/\log\log n\big)$.

For the proof of (i) and (iii) we only require that a be a constant. However to prove (ii) we need to allow a to grow with n. In what follows we will assume only that

$$a\leqq\log\log\log\log n. \tag{3.2b}$$

Let now $Z^{(k-1)}=Z_{m_1}^{(k-1)}$ and $\hat{Z}^{(k-1)}$ be the set of vertices of degree $k-1$ in the graph $H'=(V_n,\{e_u,e_{u+1},\dots,e_{m_1}\})$. Note that H' has the same distribution as G_{m_1-u+1} and we may use Lemma 1 (a) with $\omega=\log\log\log n+2a+O(1/n)$. (The $O(1/n)$ term accounts for ω_n integral.)

G_{m_1} is obtained from H' by adding $u-1$ random edges and so $Z^{(k-1)}\subseteq\hat{Z}^{(k-1)}$. By applying (2.2c) twice: once with $\omega=\log\log\log n$ and $\varepsilon=\omega^{-1}$ and once with $\omega=\log\log\log n+2a$, ε as before, we obtain

$$\Pr\left(\frac{1-\varepsilon}{1+\varepsilon}\leqq e^{2\alpha}\frac{|Z^{(k-1)}|}{|\hat{Z}^{(k-1)}|}\leqq\frac{1+\varepsilon}{1-\varepsilon}\right)\geqq 1-\frac{2\alpha(\log\log\log n)^2}{\log\log n}. \tag{3.3}$$

Now let $G'_m=(V_n,E'_m)$ where $E'_m=E_m-E_{u-1}$ and note that G'_m has the same distribution as G_{m-u+1}. Let $\hat{m}=\min\{m\colon\delta(G'_m)\geqq k\}$, and let $\hat{v}$ be the unique ($[P=1-o(1)]$, see (2.8)) vertex of degree $k-1$ in $G'_{\hat{m}-1}$. Then

$$\delta(H_u)=k\leftrightarrow\hat{v}\in Z^{(k-1)}.$$

Now $\hat{v}$ is "close to" being a random element of $\hat{Z}^{(k-1)}$ and so we can see from (3.3) that $\Pr(\hat{v}\in Z^{(k-1)})\approx e^{-2a}$, but let us do this more carefully.

Consider now the set $\{e_{m_1},e_{m_1+1},\dots,e_N\}$ and in particular the subset F of edges incident with one vertex in $\hat{Z}^{(k-1)}$ and one vertex not in $\hat{Z}^{(k-1)}$. We know ($[P=1-o(1)$, see (2.8)) that $e_{\hat{m}}\in F$. For $v\in\hat{Z}^{(k-1)}$ let d_v be the degree of v in G_{m_1}. We work on the assumption that $\hat{Z}^{(k-1)}$ is an independent set in G_{m_1}, $[P=1-o(1)]$. Now $d_v=k-1$ for $v\in Z^{(k-1)}$ and $d_v\leqq 6\log\log n$ otherwise ($[P=1-o(1)]$, see (2.9)). If d_v were the same for all $v\in\hat{Z}^{(k-1)}$ then we could deduce that $\Pr(\hat{v}\in Z^{(k-1)})=|Z^{(k-1)}|/|\hat{Z}^{(k-1)}|$ and we would be done. This is nearly so and for each $v\in\hat{Z}^{(k-1)}$

we randomly select a set $F_v \subseteqq F$ of $n_1 = \lceil n-1-6\log\log n \rceil$ edges incident with v. Let $F' = \bigcup_{v \in Z^{(k-1)}} F_v$. Then

$$\Pr(\hat{v} \in Z^{(k-1)} | e_{\hat{m}} \in F') = \frac{|Z^{(k-1)}|}{|\hat{Z}^{(k-1)}|}$$

and

$$\Pr(e_{\hat{m}} \notin F') = \frac{1}{n_1} + O\left(\frac{\log n}{n}\right).$$

$\left(\text{The } O\left(\frac{\log n}{n}\right)\right.$ term above, accounts for $\left. e_{\hat{m}} \subseteqq \hat{Z}^{(k-1)}\right)$. This completes the proof of (3.2a).

We show next that

$$\lim_{n\to\infty} \Pr(\tau'(D_k) = \tau'(C_k)) = 1. \tag{3.4}$$

Observe that if $\tau'(C_k) < u = \tau'(D_k)$ then either $\tau(D_k) \neq \tau(C_k)$ or there exists $S \subseteqq V_n$, $|S| = k-1$ such that

(3.5) S is a non-trivial separator of H_{u-1}.

Note next that (3.2) implies

$$\lim_{n\to\infty} \Pr(\tau'(D_k) \geqq n \log\log\log\log n) = 0.$$

Thus if $u_1 = \lceil n \log\log\log\log n \rceil$, $\tau'(C_k) < u < u_1$, $K = (V_n, \{e_{u_1}, e_{u_1+1}, \ldots, e_{m_1}\})$ and S is as in (3.5), then either

(a) S is a non-trivial separator of H_{u-1} and $H_{u-1}[V_n - S]$ has a component of size t, $3 \leqq t \leqq \frac{1}{2}|V_n - S|$,

or

(b) S is a non-trivial separator of K

or

(c) $\delta(K) \leqq k-2$,

or

(d) 2 vertices of K of degree $k-1$ share a common neighbour.

But K has the same distribution as $G_{m_1 - u_1 + 1}$ and (2.2) shows that these 4 events all have probability tending to zero.

(i) and (iii) follow directly from (3.1) and (3.4).

(ii).

We use

$$E\left(\frac{1}{n}\tau'(\Pi)\right) = \int_0^{(1/2)n} \Pr(\tau'(\Pi) \geqq nx)\,dx$$

and

$$\tau'(D_k) \geqq \tau'(C_k) \quad \text{whenever} \quad \tau(D_k) = \tau(C_k).$$

Let $\lambda = \log\log\log\log n$ and $\mathscr{E}$ denote the event "$\tau(D_k) = \tau(C_k)$". Then

$$\Pr(\bar{\mathscr{E}}) = O\left(\frac{(\log n)^2}{\sqrt{n}}\right). \tag{3.6}$$

This follows from (2.2d) and the proof of Theorem VII. 4 of [2]. Now

$$\begin{aligned} E\left(\frac{1}{n}\tau'(C_k)\right) &\geqq \int_0^\lambda \Pr(\tau'(C_k) \geqq nx)\,dx \geqq \\ &\geqq \int_0^\lambda e^{-2x}\,dx - O\left(\frac{\lambda(\log\log\log n)^2}{\log\log n}\right) = \quad \text{by (3.2)} \\ &= \frac{1}{2} - o(1). \end{aligned} \tag{3.7}$$

Now a lower bound for $E(\tau'(D_k))$.

$$\begin{aligned} E\left(\frac{1}{n}\tau'(D_k)\right) &= E\left(\frac{1}{n}\tau'(D_k)|\mathscr{E}\right)\Pr(\mathscr{E}) \geqq \\ &\geqq E\left(\frac{1}{n}\tau'(C_k)|\mathscr{E}\right)\Pr(\mathscr{E}) = \\ &= E\left(\frac{1}{n}\tau'(C_k)\right) - E\left(\frac{1}{n}\tau'(C_k)|\bar{\mathscr{E}}\right)\Pr(\bar{\mathscr{E}}) \geqq \\ &\geqq \frac{1}{2} - o(1) - E\left(\frac{1}{n}\tau(C_k)|\bar{\mathscr{E}}\right)\Pr(\bar{\mathscr{E}}). \end{aligned}$$

But

$$E\left(\frac{1}{n}\tau(C_k)|\bar{\mathscr{E}}\right) \leqq (\log n)^2 + n\Pr\left(\tau(C_k) \geqq n(\log n)^2\right)/\Pr(\bar{\mathscr{E}}).$$

Thus, using (3.6) and $\Pr(\tau(C_k) \geqq n(\log n)^2) = o(1/n)$ (very crudely), we have

$$E\left(\frac{1}{n}\tau(C_k)|\bar{\mathscr{E}}\right)\Pr(\bar{\mathscr{E}}) = o(1) \tag{3.8}$$

and so

$$E\left(\frac{1}{n}\tau'(D_k)\right) \geqq \frac{1}{2} - o(1).$$

Now for upper bounds:

$$\begin{aligned} E\left(\frac{1}{n}\tau'(D_k)\right) &= \int_0^\lambda \Pr(\tau'(D_k) \geqq nx)\,dx + \int_\lambda^{(1/2)n} \Pr(\tau'(D_k) \geqq nx)\,dx \leqq \\ &\leqq \int_0^\lambda e^{-2x}\,dx + O\left(\frac{\lambda(\log\log\log n)^2}{\log\log n}\right) + \int_\lambda^\infty 2\alpha e^{-x/2}\,dx + \int_\lambda^{(1/2)n} n^{-2}\,dx, \\ &= \frac{1}{2} + o(1). \end{aligned} \tag{3.9}$$

The first integral in (3.9) is approximated as in (3.7). For the second note that

$$\Pr\big(\tau'(D_k) \geqq nx\big) \leqq \Pr\big(\delta(G_{m^+}) < k\big) + \Pr\big(\delta(G_{m^-}) \geqq k\big) \tag{3.10}$$

where

$$m^+ = \frac{1}{2} n \log n + \frac{1}{2}(k-1) n \log\log n + \frac{1}{2} nx$$

and

$$m^- = m^+ - nx.$$

Now use (2.3) with $m=m^-$ and (2.4) with $m=m^+$ in (3.10). Now an upper bound for $E\big(\tau'(c_k)\big)$.

$$E\left(\frac{1}{n}\tau'(C_k)\right) = E\left(\frac{1}{n}\tau'(C_k)|\mathscr{E}\right)\Pr(\mathscr{E}) + E\left(\frac{1}{n}\tau'(C_k)|\bar{\mathscr{E}}\right)\Pr(\bar{\mathscr{E}}) \leqq$$

$$\leqq E\left(\frac{1}{n}\tau'(D_k)|\mathscr{E}\right)\Pr(\mathscr{E}) + E\left(\frac{1}{n}\tau(C_k)|\bar{\mathscr{E}}\right)\Pr(\bar{\mathscr{E}}) \leqq$$

$$\leqq E\left(\frac{1}{n}\tau'(D_k)\right) + o(1) \leqq \quad \text{(by (3.8))}$$

$$\leqq \frac{1}{2} + o(1) \quad \big(\text{by (3.10)}\big).$$

This completes the proof of our theorem.

4. Revival Time

We gain some insight into the distribution of survival time by considering the *revival time*. Let $u=\tau'(\Pi)$ and consider adding random edges to H_u until a graph with property Π is obtained. The number of edges added $\tau''(\Pi)$ is called the revival time. It is straightforward to show that we can replace τ' by τ'' in our theorem and obtain a valid result.

For example: if $\Pi=D_k$ then $H_u[P=1-o(1)]$ contains a unique vertex v of minimal degree $k-1$. $\tau''\geqq an$ if v is not incident with any of the first an random edges added to H_u. The probability of this is approximately $\left(1-\frac{2}{n}\right)^{an} \approx e^{-2a}$. For $\Pi=C_k$ we use the smaller likelihood of non-trivial separators.

It seems now that in general one may be able to guess the result for survival time by computing the revival time, which seems easier. One then needs a few asymptotic calculations for verification.

5. Open Questions

What is the survival time for the following properties:

(1) having a cycle?
(2) having a cycle of size k?
(3) having a path of length k?
(4) having a tree component of size k?
(5) having a clique of size k?
(6) having any fixed subgraph?
(7) being non-planar?
(8) having diameter k?
(9) having a vertex of degree k?

One can also consider similar problems for random bipartite graphs, digraphs or subgraphs of the n-cube.

There are 2 conspicuous omissions from our list. These are perfect matchings and hamilton cycles. If H_k is the property of having $\lfloor k/2 \rfloor$ edge disjoint hamilton cycles plus a further edge disjoint matching of size $\lfloor n/2 \rfloor$, if k is odd, then it seems fairly clear that we can add H_k to our theorem. The proof does not require any new ideas and would be rather long, too long for this paper.

Appendix

As usual let G_p, $p=\frac{N}{m}$ denote the random graph in which edges are independently included with probability p. Let Π be any graph property. Then

$$\text{(A0)} \qquad \Pr(G_p \in \Pi) = \sum_{m'} \Pr(G_{m'} \in \Pi) \Pr(G_p \text{ has } m' \text{ edges}).$$

Thus if Π is monotone i.e. if it is preserved either by adding edges or by deleting edges, then, for large n

$$\text{(A1)} \qquad \Pr(G_m \in \Pi) \leqq 3 \Pr(G_p \in \Pi).$$

We can thus work mainly with G_p and multiply our estimates by 3. Our inequalities are only required to hold for n large.

Proof of (2.2a).
For $k \geqq 2$

$$\Pr\big(\delta(G_p) \leqq k-2\big) \leqq n \sum_{t=0}^{k-2} \binom{n-1}{t} p^t (1-p)^{n-1-t} \leqq$$

$$\leqq 2n \sum_{t=0}^{k-2} \frac{n^t}{t!} \left(\frac{\log n}{n}\right)^t \frac{(\log n)^{-(k-1)} e^{\omega}}{n} \leqq$$

$$\leqq \frac{3e^{\omega}}{(k-2)! \log n}$$

Now use (A1).

Proof of (2.2b).

Let $z_p^{(k-1)}$ be the number of vertices of degree $k-1$ in G_p.

$$E(z_p^{(k-1)}) = n\binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} =$$
$$= \left(1+O\left(\frac{(\log n)^2}{n}\right)\right)\left(\frac{np}{\log n}\right)^{k-1}\frac{e^{\omega}}{(k-1)!} =$$
$$= \left(1+O\left(\frac{\log\log n}{\log n}\right)\right)\frac{e^{\omega}}{(k-1)!}.$$

Preparing for the Chebyshef inequality

$$E\big(z_p^{(k-1)}(z_p^{(k-1)}-1)\big) =$$
$$= n(n-1)\left(p\left(\binom{n-2}{k-2}p^{k-2}(1-p)^{n-k}\right)^2+(1-p)\left(\binom{n-2}{k-1}p^{k-1}(1-p)^{n-k-1}\right)^2\right) =$$
$$= n(n-1)\left(p\left(\frac{E(z_p^{(k-1)})}{n(n-1)p}\right)^2+(1-p)\left(\frac{E(z_p^{(k-1)})}{n(1-p)}\,\frac{n-1}{n-k}\right)^2\right) =$$
$$\leqq (1+2p)\big(E(z_p^{(k-1)})^2\big)$$

and so

$$\text{(A2)}\qquad \operatorname{Var}(z_p^{(k-1)}) \leqq E(z_p^{(k-1)})+2pE(z_p^{(k-1)})^2 \leqq$$
$$\leqq 2E(z_p^{(k-1)}) \quad \text{as } \omega = o(\log\log n).$$

Thus by the Chebycheff inequality

$$\Pr(z_p^{(k-1)}=0) \leqq \frac{2}{E(z_p^{(k-1)})} \leqq 3(k-1)!\,e^{-\omega}.$$

Thus

$$\Pr\big(\delta(G_p) \geqq k\big) \leqq 3(k-1)!\,e^{-\omega}.$$

We can now use (A1).

Proof of (2.2c).

The property in question is not monotone (or even convex) [2] and so we cannot use (A1). We can however assert using (A2) that

$$\Pr\left(\left|z_p^{(k-1)}-\frac{e^{\omega}}{(k-1)!}\right| \geqq \frac{\varepsilon e^{\omega}}{(k-1)!}\right) \leqq \frac{2e^{-\omega}}{\varepsilon^2}.$$

It now follows from (A0) that there exists $m', m-\sqrt{n\log n} \leqq n' \leqq n$, such that

$$\Pr\left(\left|z_{m'}^{(k-1)}-\frac{e^{\omega}}{(k-1)!}\right| \geqq \frac{\varepsilon e^{\omega}}{(k-1)!}\right) \leqq \frac{2e^{-\omega}}{\varepsilon^2}.$$

As in the proof of (2.2a) we can deduce that

$$\Pr(\delta(G_{m'}) < k-1) \leq \frac{10e^{\omega}}{(k-2)!\log n}.$$

Since G_m is obtained from $G_{m-m'}$ by adding $m-m'$ random edges, we have that given $\mathscr{E} =$"$\delta(G_{m'}) \geq k-1$ and $z_{m'}^{(k-1)}$ is 'close' to its mean",

$$\Pr(z_m^{(k-1)} \neq z_{m'}^{(k-1)} | \mathscr{E}) = O\left(\frac{\sqrt{n \log n}\, ne^{\omega}}{N-n}\right) =$$

$$= O\left(\sqrt{\frac{\log n}{n}}\, e^{\omega}\right)$$

and (2.2c) follows.

Proof of (2.2d).

Let θ be the probability that G_p has a non-trivial separator of size s, $0 \leq s \leq$ $\leq k-1$. Then

$$\theta \leq \sum_{s=0}^{k-1} \sum_{t=2}^{(1/2)(n-s)} \binom{n}{s} \binom{n-s}{t} t^{t-2} p^{t-1} (1-(1-p)^t)^s (1-p)^{t(n-s-t)}$$

(choose an s-set S for the separator, a t-set T for a small component, a spanning tree of T. Multiply by the probability that the edges of the tree exist and there is at least one v, T edge for each $v \in S$ and no S, $S \cup T$ edges.)

Thus

$$\theta \leq \sum_{s=0}^{k-1} \sum_{t=2}^{(1/2)(n-s)} \left(\frac{ne}{s}\right)^s \left(\frac{ne}{t}\right)^t t^{t-2} p^{t-1} (tp)^s e^{-t(n-s-t)p} \leq$$

(A3) $$\leq 8 \sum_{s=0}^{k-1} \sum_{t=2}^{(1/2)(n-s)} \frac{1}{p} (npe^{-np} t^{s-2/t} e^{(s+t+1)p})^t =$$

$$= O\left(\frac{\log n}{n}\right)$$

(for small t, the "complex" term above is $O\left(\left(\frac{e^{\omega} \log n}{n}\right)^t\right)$. For larger t, it is $O\left(\left(\frac{e^{\omega} \log n}{\sqrt{n}}\right)^t\right)$. Unfortunately, we are not dealing with a monotone property. However (A0) implies

$$\Pr(G_m \text{ has a non-trivial separator}) \leq \theta \Pr(G_p \text{ has } m \text{ edges})^{-1}$$

and (2.2d) follows. (See [2] Theorem II.2.)

Proof of (2.2e).

We only have to consider the sum in (A3) for $t \geq 3$, which is then $O\left(\left(\frac{\log n}{n}\right)^2\right)$. We only have to multiply this by $O\left(n \log\log n \times \frac{\sqrt{n}}{\sqrt{\log n}}\right)$ in order to account for the number of different values of m and the transition from G_p to G_m.

Proof of (2.2f).

We can clearly assume $k \geqq 2$. Let ψ be the probability that there exist $v, w \in Z_m^{(k-1)}$ at distance 2 or less from each other in G_p. Then

$$\psi \leqq \sum_{r=2}^{3} \frac{r!}{2} \binom{n}{r} p^{r-1} \left(\binom{n-r}{k-2} p^{k-2} (1-p)^{n-k-r+2} \right)^2 +$$

$$+3 \binom{n}{3} p^3 \left(\binom{n-3}{k-2} p^{k-2} (1-p)^{n-k-1} \right)^2$$

(the first term above deals with paths of length $r=2$ or 3 and the second term deals with triangles.) Thus $\psi = O\left(\frac{1}{n}\right)$ and we can finish as in the proof of (2.2d).

Proof of (2.3).

The proof used for (2.2b) will be valid for $p \geqq p_0 = \frac{\log n}{4n}$, say. All that is needed for smaller p is to show

$$\Pr(G_{p_0} \text{ has no isolated vertices}) = O\left(\frac{e^{np}}{n}\right).$$

This can be done using the Chebycheff inequality.

Proof of (2.4).

Non-trivial separators are handled as in the proof of (2.2d). The minimum degree calculation only requires the use of the expected number of vertices of degree $k-1$ or less. ∎

References

[1] B. Bollobás, The evolution of sparse graphs, In *Graph Theory and Combinatorics*, Proc. Cambridge Combinatorial Conference in houour of Paul Erdős (B. Bollobás, Ed.), Academic Press (1984), 35—57.

[2] B. Bollobás, *Random Graphs*, Academic Press, 1985.

[3] B. Bollobás and A. Thomason, Random graphs of small order, *Annals of Discrete Mathematics*.

[4] P. Erdős and A. Rényi, On random Graphs I, *Publ. Math. Debrecen*, 6 (1959), 290—297.

[5] P. Erdős and A. Rényi, On the evolution of random graphs, *Publ. Math. Inst. Hungar. Acad. Sci.*, 7 (1960), 17—61.

[6] P. Erdős and A. Rényi, On the strength of connectedness of a random graph, *Acta Math. Acad. Sci. Hungar.*, 12 (1961), 261—267.

[7] E. Palmer, *Graphical Evolution.*

A. M. Frieze

Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213